Econ 802

Answers to Final Exam

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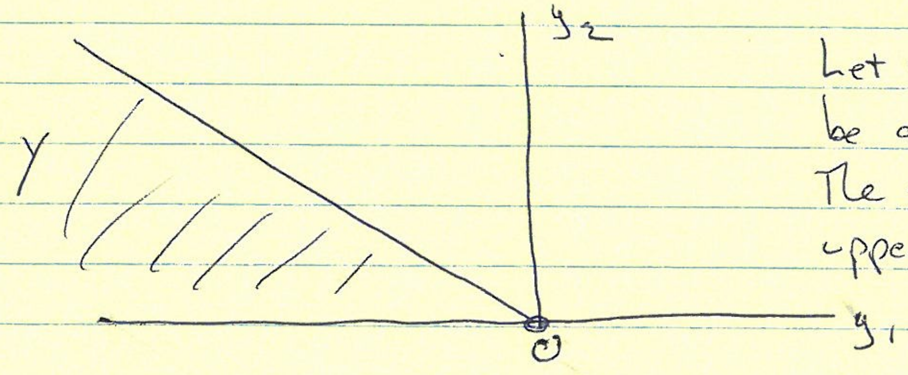
1. (a) Consider two price vectors p and p' with $p' \geq p$, where these vectors only differ for the prices of the outputs. Let y be optimal for p and let y' be optimal for p', so $\pi(p') = p'y' \geq p'y$ and $\pi(p) = py \geq py'$. We also have $p'y \geq py$ because output quantities are positive and $p_i' \geq p_i$ for all $y_i > 0$. Therefore $\pi(p') = p'y' \geq p'y \geq py = \pi(p)$.

(b) Y has CRS if $y \in Y$ implies $ty \in Y$ for all $t > 0$. In this case the profit function is not well-defined if there is any production plan that gives positive profit. Suppose $y_0 \in Y$ and $py_0 > 0$. Then we can also choose any ty_0 and get profit $t(py_0)$. Since t is unbounded profit is unbounded. If the firm cannot get positive profit and the origin is in Y, then maximum profit is zero and $\pi(p) = 0$ is well-defined.

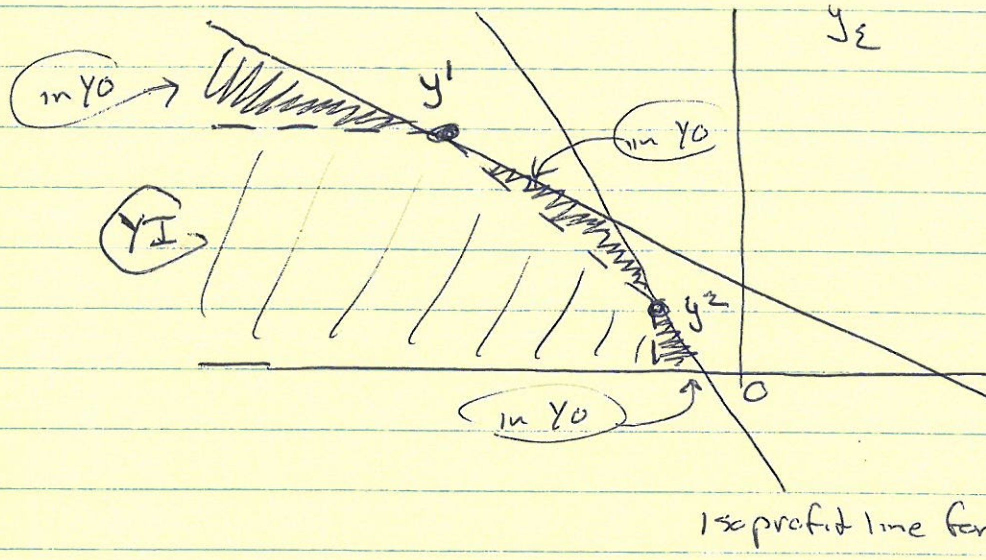


Let y_1 be an input and y_2 be an output. The set Y has a linear upper boundary and thus has CRS

An isoprofit line has $\pi = p_1y_1 + p_2y_2$ or $y_2 = \frac{\pi - p_1y_1}{p_2}$

If the isoprofit lines are steeper than the upper boundary of Y then the maximum profit is zero (the origin is optimal). If the isoprofit lines are flatter, positive profit is feasible and $\pi(p)$ is undefined.

1(c). Let p^1 and p^2 be the price vectors at $t = 1, 2$ and let y^1 and y^2 be the production plans observed. Due to WAPM, $p^1y^1 \geq p^1y^2$ and $p^2y^2 \geq p^2y^1$.



Suppose the observations look like this (note that this is consistent with WAPM)

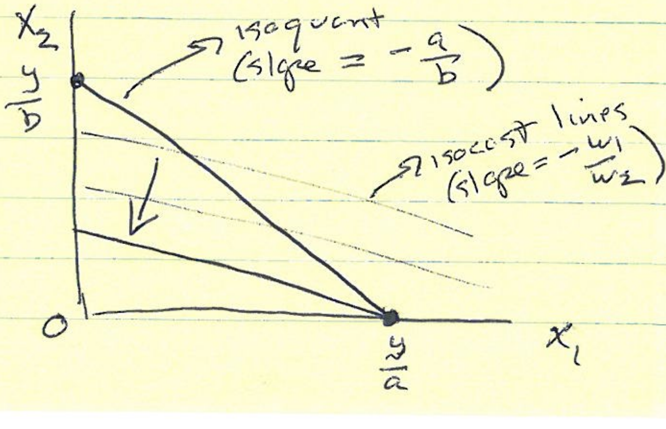
YI is the smallest set that is monotonic, convex, and consistent with the observations (it includes only the shaded area on and below the dashed lines)

YO is the largest set that is monotonic, convex, and consistent with the observations (it includes every point on or below both of the isoprofit lines)

These sets are important because they place bounds on the true production possibilities set Y. Also, they show that whenever WAPM holds there is some technology that can account for the firm's behavior in a way that is consistent with profit maximization.

2. (a) ~~$\min [w_1x_1 + w_2x_2]$~~

Consider the case where $\frac{w_1}{w_2} < \frac{a}{b}$ so the isocost lines are flatter than the isoquant and the solution is $x_1 = \frac{y}{a}$, $x_2 = 0$.



$$L = -w_1 x_1 - w_2 x_2 + \lambda[a x_1 + b x_2 - y] + \mu_1 x_1 + \mu_2 x_2$$

Note that we are minimizing cost, so I multiplied the objective $w_1 x_1 + w_2 x_2$ by -1 to convert it into a max problem.

The FOC are:
$$\frac{\partial L}{\partial x_1} = -w_1 + \lambda a + \mu_1 = 0$$
$$\frac{\partial L}{\partial x_2} = -w_2 + \lambda b + \mu_2 = 0$$

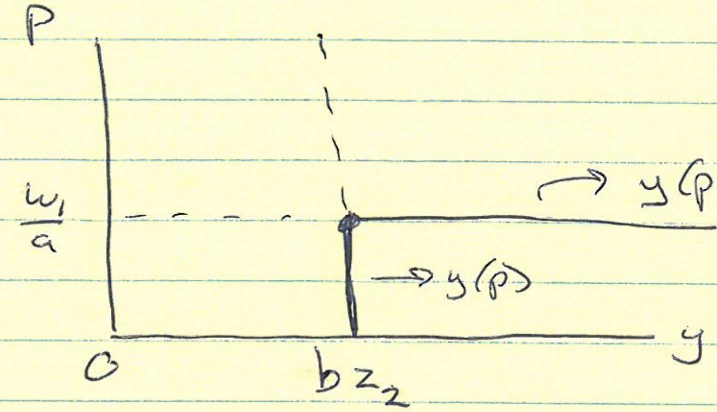
$$\lambda \geq 0, \quad a x_1 + b x_2 - y \geq 0, \quad \lambda[a x_1 + b x_2 - y] = 0$$
$$\mu_1 \geq 0, \quad x_1 \geq 0, \quad \mu_1 x_1 = 0; \quad \mu_2 \geq 0, \quad x_2 \geq 0, \quad \mu_2 x_2 = 0.$$

To verify that the corner solution from the graph satisfies these conditions, start with $x_1 = \frac{y}{a} > 0$ which implies $\mu_1 = 0$. This gives $-w_1 + \lambda a = 0$ or $\lambda = \frac{w_1}{a} \geq 0$. Therefore, we have $a x_1 + b x_2 - y = 0$, which is true. Because $x_2 = 0$ we have $\mu_2 x_2 = 0$. Finally $\mu_2 = w_2 - \lambda b = w_2 - \frac{b w_1}{a} > 0$ where the positive sign follows from the assumption about the relative slopes of isoquants and isocost lines. So all FOC hold.

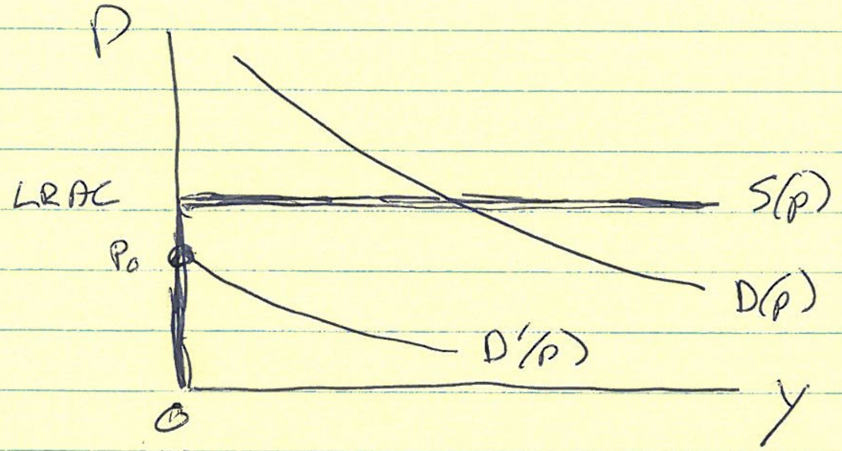
(b) The firm produces $b z_2$ no matter what the output price p is. It uses x_1 when p is high enough. Write profit as
$$p y - w_1 x_1 - w_2 x_2 = p[a x_1 + b x_2] - w_1 x_1 - w_2 x_2$$
$$= (pa - w_1) x_1 + (pb - w_2) z_2$$
If $p < \frac{w_1}{a}$, the firm sets $x_1 = 0$
if $p = \frac{w_1}{a}$, the firm is indifferent toward x_1
if $p > \frac{w_1}{a}$, the profit max problem has no solution.

So mathematically the SR supply function is
$y(p) = b z_2$ for $0 \leq p < \frac{w_1}{a}$; $y(p) \geq b z_2$ at $p = \frac{w_1}{a}$;
$y(p)$ undefined otherwise.

Graphically:



2(c) The production function has CRS so long run average cost is a constant. The fixed number of firms is not important because when $P = LRAC$ each firm is willing to produce an unlimited quantity.



The LR supply curve $S(p)$ has a vertical segment up to LRAC and then becomes horizontal. If $D(p)$ intersects the horizontal part of $S(p)$, there is a unique equilibrium price $= LRAC$.

If we have a market demand like D' with a vertical intercept p_0 below LRAC, there is an equilibrium price (supply = demand = 0 for any $p \in [p_0, LRAC]$) but it is not unique. The case where $D(p)$ is always above $S(p)$ can be ignored because it requires a violation of the consumers' budget constraints.

3(a) In order for x^* to solve the utility max problem, it must satisfy the FOC. These are

$$\frac{\partial u(x)}{\partial x_i} = \lambda p_i \text{ for all } i = 1 \ldots n \text{ and } px = 1$$

Multiply each FOC for good i by x_i to get

$\frac{\partial u(x)}{\partial x_i} x_i = \lambda p_i x_i$ and then sum over goods to get

$$\sum_i \frac{\partial u(x)}{\partial x_i} x_i = \lambda \text{ where } \sum_i p_i x_i = px = 1.$$

This gives $\dfrac{\frac{\partial u(x)}{\partial x_i}}{\lambda} = \dfrac{\frac{\partial u(x)}{\partial x_i}}{\sum_i \frac{\partial u(x)}{\partial x_i} x_i} = p_i$ for all i.

Substitute x^* in these conditions to obtain p^*. At these prices, x^* satisfies the FOC for utility max. The FOC are sufficient due to strict quasi-concavity.

(b) This result occurs because $v(p, m)$ is homogeneous of degree zero in (p, m). If all prices and income are multiplied by the same scalar $t > 0$, the set of feasible consumption bundles does not change, so the most preferred bundle in the feasible set does not change, and the resulting utility does not change.

Mathematically: $v(tp, tm) \equiv v(p, m)$ for all $t > 0$. Differentiate both sides with respect to t to get

$\sum_i \frac{\partial v(tp, tm)}{\partial p_i} p_i + \frac{\partial v(tp, tm)}{\partial m} m = 0$, then evaluate

everything at $t = 1$ and reorganize the equation.

3(c) The easiest method is to work with the aggregate indirect utility function $V(p, M) = \sum_i a_i(p) + b(p) M$. To keep utility constant, we need

$$V(p, M) = V(p', M') \quad \text{or}$$

$$\sum_i a_i(p) + b(p) M = \sum_i a_i(p') + b(p') M'$$

$$\Rightarrow M' = \frac{1}{b(p')}\left[\sum_i a_i(p) - \sum_i a_i(p') + b(p) M\right]$$

Another method that also works (although it is more roundabout) is to solve for the expenditure functions of the individual consumers, sum them up to get the aggregate expenditure function, and use that to compute M'. This should give the same answer.

4(a) Let's start with the contract curve. The indifference curves for A and B have a tangency when $MRS_A = MRS_B$ or

$$\frac{MU_{A1}}{MU_{A2}} = \frac{MU_{B1}}{MU_{B2}}$$

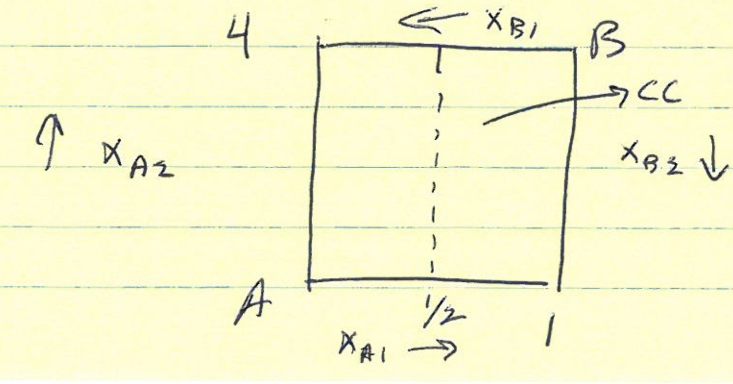
where $MU_{A1} = \frac{1}{x_{A1}^2}$ $\quad MU_{B1} = \frac{1}{x_{B1}^2}$

$MU_{A2} = 1$ $\quad MU_{B2} = 1$

So a point on the contract curve has

$$\frac{1}{x_{A1}^2} = \frac{1}{x_{B1}^2} \Rightarrow x_{A1} = x_{B1}$$

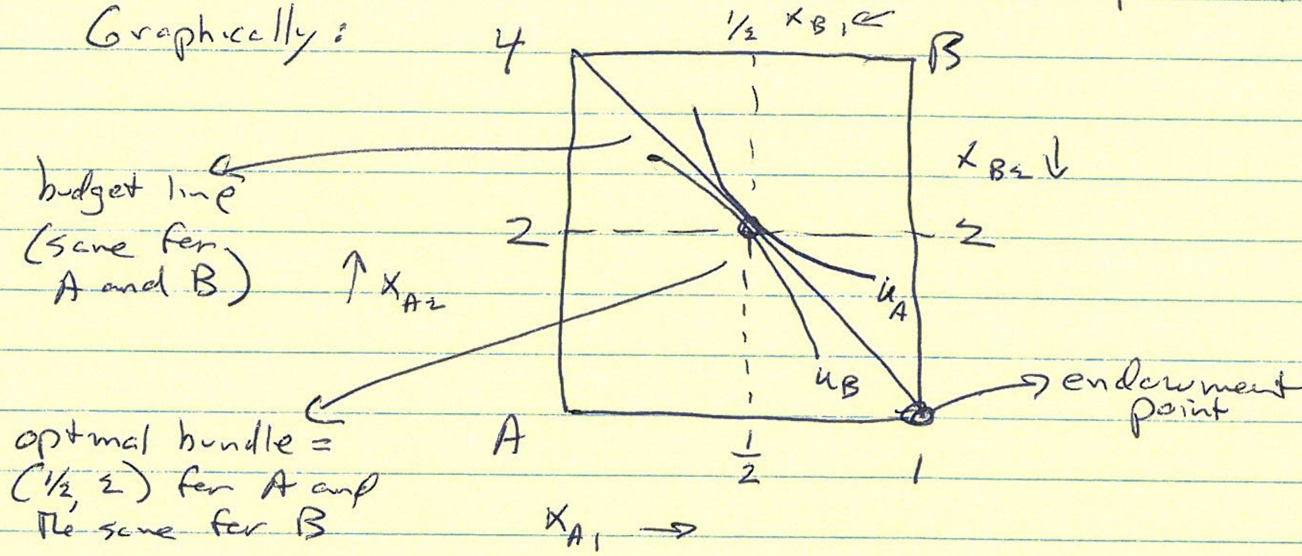
Thus a point is on CC if and only if good 1 is divided equally:



This is true along the vertical dashed line at 1/2. Interpretation: both A and B have quasi-linear utility so there are no income effects for good 1.

4(b) Some quick intuition: we know from the first welfare theorem that any equilibrium allocation will be on the contract curve, and we know that the equilibrium price ratio will equal MRS_A and MRS_B. Using $x_{A1} = x_{B1} = 1/2$ we have $MRS_A = MRS_B = 4 = \frac{p_1}{p_2}$. This clears the market for good 1 by construction. From Walras's Law it must also clear the market for good 2. Note by homogeneity of aggregate excess demand we can only solve for the price ratio, not the absolute level of the individual prices.

Graphically:



(c) $$L = -\frac{1}{x_{A1}} + x_{A2} - \frac{1}{x_{B1}} + x_{B2} - \lambda_1\left[x_{A1} + x_{B1} - 1\right] - \lambda_2\left[x_{A2} + x_{B2} - 4\right]$$

$$\frac{\partial L}{\partial x_{A1}} = \frac{1}{x_{A1}^2} - \lambda_1 = 0 \qquad \frac{\partial L}{\partial x_{B1}} = \frac{1}{x_{B1}^2} - \lambda_1 = 0$$

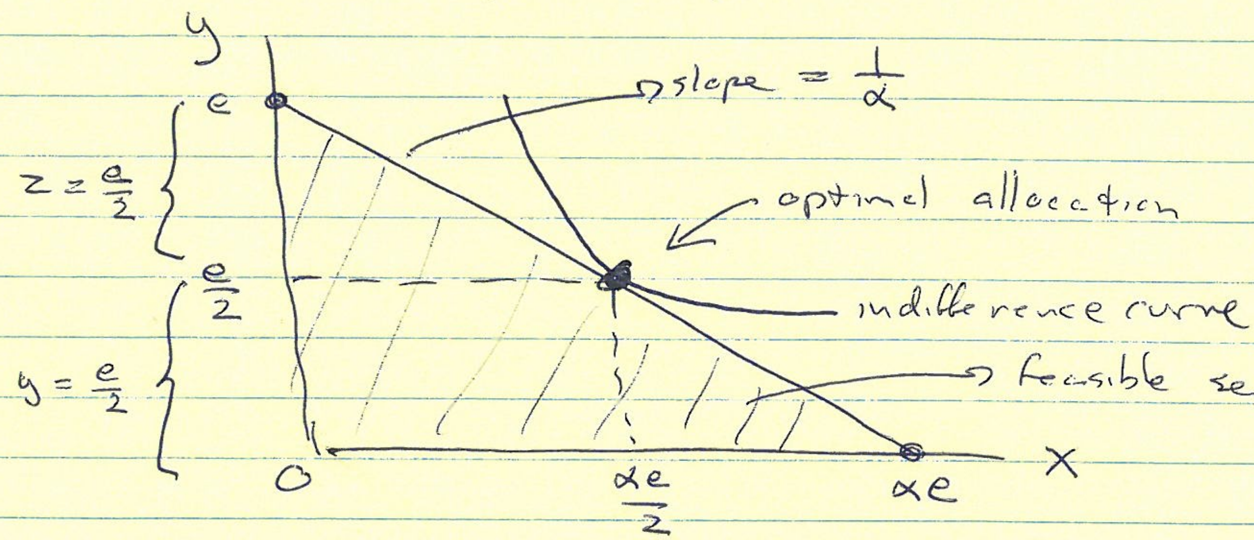
$$\frac{\partial L}{\partial x_{A2}} = 1 - \lambda_2 = 0 \qquad \frac{\partial L}{\partial x_{B2}} = 1 - \lambda_2 = 0$$

This gives $\frac{1}{x_{A1}^2} = \frac{1}{x_{B1}^2} = \lambda_1$ so $x_{A1} = x_{B1}$ as before. Thus the planner divides good 1 equally. However there is no unique solution for the allocation of good 2. Due to quasi-linearity, redistributing good 2 from A to B does not affect the sum of the utilities.

5 (a) If Crusoe consumes x, he can have $y = e - \frac{x}{\alpha}$ because $z = \frac{x}{\alpha}$ of his endowment e is used up. To max utility, we max $\ln x + \ln\left(e - \frac{x}{\alpha}\right)$

FOC: $\frac{1}{x} + \frac{1}{\left(e - \frac{x}{\alpha}\right)}\left(-\frac{1}{\alpha}\right) = 0$

$\Rightarrow \quad x = \frac{\alpha e}{2}, \quad y = \frac{e}{2}, \quad z = \frac{e}{2}$



(b) The firm's profit is $px - qz = p\alpha z - qz = z(p\alpha - q)$
Because the production function is linear, we have CRS so we need the max profit to be zero in order for the profit maximization problem to have a solution.
This is true when $p\alpha - q = 0$ or $\boxed{\frac{p}{q} = \frac{1}{\alpha}}$
The firm is willing to produce any quantity at this price ratio. Crusoe's budget constraint is $px + qy = qe$; his Lagrangian is
$L = \ln x + \ln y - \lambda[px + qy - qe]$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= \frac{1}{x} - \lambda p = 0 \\ \frac{\partial L}{\partial y} &= \frac{1}{y} - \lambda q = 0 \end{aligned}\right\} \Rightarrow \frac{y}{x} = \frac{p}{q} = \frac{1}{\alpha}$$

which gives the same consumption bundle as in part (a)

[Note that the budget constraint is the same as the boundary of the feasible set in the graph.]

To show that both markets clear, observe that Crusoe demands $\frac{\alpha e}{2}$ of the x good, which the firm is happy to supply (it gets zero profit for any output); and Crusoe demands $y = \frac{e}{2}$ while the firm demands $z = \frac{e}{2}$ as an input, which adds up to the supply e from Crusoe's endowment.

(c) Continue to use $\frac{p}{q} = \frac{1}{\alpha}$. For the same reasons as in (b), each firm's maximum profit is zero and it can achieve this profit for any non-negative output. Each consumer i has the budget constraint

$$p x_i + q y_i = qe \quad \text{and utility } u_i = \ln x_i + \ln y_i.$$

As before, the FOC give $\frac{y_i}{x_i} = \frac{1}{\alpha} = \frac{p}{q}$

so $x_i = \alpha y_i \Rightarrow$

$$p\alpha y_i + q y_i = qe \Rightarrow y_i\left[p\alpha + q\right] = qe$$

$$\Rightarrow \begin{cases} y_i = \dfrac{qe}{p\alpha + q} = \dfrac{e}{\frac{p}{q}\alpha + 1} = \dfrac{e}{2} \\ x_i = \dfrac{\alpha e}{2} \end{cases}$$

Note: I am ignoring the T_{ij} because the firms will all have zero profit.

Total demand for the x good is

$X = \frac{n\alpha e}{2}$ which can be satisfied by having the firms produce the same total output (so the market for the x good clears)

Total demand for the y good from the firms is

$$Z = \frac{X}{\alpha} = \frac{ne}{2}$$

Total demand for the y good from all the Crusoes is

$$Y = \frac{ne}{2}$$

The market for the y good clears because $\overbrace{Y + Z}^{\text{demand}} = \underset{\text{supply}}{E}$ where $E = ne$ is the aggregate endowment of the y good.